Comments on "Uniformly Loaded Plates of Regular Polygonal Shape"

H. D. CONWAY*
Cornell University, Ithaca, N.Y.

It should be mentioned that point-matching results for certain uniformly loaded, polygonal plates with clamped[2] and simply supported[3] edges are already available in the literature. The moments at the centers of uniformly loaded, simply supported plates in the form of regular polygons are most conveniently obtained by solving the membrane-type equation

$$\nabla^2 M = -q \qquad M = (M_x + M_y)/(1 + \nu)$$

subject to $M = 0$ on the boundary and noting[2] that $M_x = M_y = (1 + \nu)M/2$ at the centers. Since the preceding plate problem is analogous to the torsion of a polygonal rod, for which point-matching solutions are available, the central moments in the plate are readily obtained. It may also be mentioned that the maximum deflections in membranes with invariant tension may be written down by analogy from the third column of the authors' Table 2 of Ref. 1.

References

[1] Leissa, A. W., Lo, C. C., and Niedenfuhr, F. W., "Uniformly loaded plates of regular polygonal shape," AIAA J. **3**, 566–567 (1965).

[2] Conway, H. D. "The approximate analysis of certain boundary-valve problems," J. Appl. Mech. **27**, 275–277 (1960).

[3] Conway, H. D., "The bending buckling and flexural vibration of simply supported polygon plates by point-matching," J. Appl. Mech. **28**, 288–291 (1961).

Received July 15, 1965.

* Professor of mechanics.

Comments on "A Variational Method for Optimal Staging"

WALTER F. DENHAM*
Analytical Mechanics Associates, Inc., Westbury, N. Y.

In Ref. 1, appearing in this issue, the authors treat variational problems with state variable discontinuities. They employ the method introduced by Denbow[2] and used again by Hunt and Andrus[3] that uses transformations to and back from an expanded set of variables in order to reduce the problem to Bliss' classical form. Although there is no objection to this in principle, and certainly not to the results, this comment is intended to show that a simple extension of the classical form makes the double transformation unnecessary and thus greatly simplifies the derivation.

The problem considered here is the derivation of necessary conditions for variable discontinuities (the case of fixed discontinuities is a simple special case). The performance index is given by (1),† the constraints by (2) and (3). With definitions of F and G given following (50), the Lagrange variable approach to the problem is to find $y_i(t)$, $\lambda_\beta(t)$, and l_μ, satisfying the constraints, which makes stationary

$$J = G + \int_{t_1}^{t_m} F dt \tag{1}$$

The simple extension to the classical theory is to allow for discontinuities at the t_j by writing (1) as

$$J = G + \int_{t_1}^{t_2^-} + \int_{t_2^+}^{t_3^-} + \ldots \int_{t_{m-1}^+}^{t_m} F dt \tag{2}$$

The variation of J (total differential or variational derivative) is then

$$dJ = \frac{\partial G}{\partial y_i(t_1)} dy_i(t_1) + \frac{\partial G}{\partial y_i(t_2^-)} dy_i(t_2^-) + \frac{\partial G}{\partial y_i(t_2^+)} dy_i(t_2^+) + \ldots + \frac{\partial G}{\partial y_i(t_m)} dy_i(t_m) + \frac{\partial G}{\partial t_1} dt_1 + \ldots + \frac{\partial G}{\partial t_m} dt_m + \int_{t_1}^{t_2^-} + \ldots + \int_{t_{m-1}^+}^{t_m} \left[\frac{\partial F}{\partial \dot{y}_i} \delta \dot{y}_i + \frac{\partial F}{\partial y_i} \delta y_i\right] dt \tag{3}$$

Integration of the $\delta\dot{y}_i$ terms by parts between, say, t_k^+ and t_{k+1}^-, gives

$$\int_{t_k^+}^{t_{k+1}^-} \frac{\partial F}{\partial \dot{y}_i} \delta\dot{y}_i dt = \frac{\partial F}{\partial \dot{y}_i} \delta y_i \Big|_{t_k^+}^{t_{k+1}^-} - \int_{t_k^+}^{t_{k+1}^-} \left[\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right)\right] \delta y_i dt \tag{4}$$

The one critical step in completing the derivation is the use of the first-order relationship

$$d[y_i(t_k^\pm)] = \delta y_i(t_k^\pm) + \dot{y}_i(t_k^\pm) dt_k \tag{5}$$

Because of state discontinuities at the t_j, differential changes in y_i will not be the same at t_k^- as at t_k^+. The derivatives $\dot{y}_i$ will also be different. Equation (5) shows that the variation of y_i at fixed time t_k, $\delta y_i(t_k)$ must be calculated according to which segment of the trajectory it is considered to be a part (i.e., whether it is evaluated in the segment ending at t_k or the one beginning at t_k).

Solving (5) for $\delta y_i(t_k^\pm)$ in terms of $dy_i(t_k^\pm)$ and $\dot{y}_i(t_k^\pm)$ and substituting into (4) thence into (3), dJ may be written as

$$dJ = \left[\frac{\partial G}{\partial y_i(t_1)} - \left(\frac{\partial F}{\partial \dot{y}_i}\right)_{t_1}\right] dy_i(t_1) + \left[\frac{\partial G}{\partial y_i(t_2^-)} + \left(\frac{\partial F}{\partial \dot{y}_i}\right)_{t_2^-}\right] dy_i(t_2^-) + \left[\frac{\partial G}{\partial y_i(t_2^+)} - \left(\frac{\partial F}{\partial \dot{y}_i}\right)_{t_2^+}\right] dy_i(t_2^+) + \ldots + \left[\frac{\partial G}{\partial y_i(t_m)} + \left(\frac{\partial F}{\partial \dot{y}_i}\right)_{t_m}\right] dy_i(t_m) + \left[\frac{\partial G}{\partial t_1} + \left(\frac{\partial F}{\partial \dot{y}_i} \dot{y}_i\right)_{t_1}\right] dt_1 + \left[\frac{\partial G}{\partial t_2} - \left(\frac{\partial F}{\partial \dot{y}_i} \dot{y}_i\right)_{t_2^-} + \left(\frac{\partial F}{\partial \dot{y}_i} y_i\right)_{t_2^+}\right] dt_2 + \ldots + \left[\frac{\partial G}{\partial t_m} - \left(\frac{\partial F}{\partial \dot{y}_i} \dot{y}_i\right)_{t_m}\right] dt_m + \int_{t_1}^{t_2^-} + \ldots + \int_{t_{m-1}^+}^{t_m} \left[-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right) + \frac{\partial F}{\partial y_i}\right] \delta y_i dt \tag{6}$$

The result (47) is obtained by setting the coefficient of δy_i equal to zero in each subinterval. The boundary conditions (48–50) are obtained by setting the coefficients of $dy_i(t_k^\pm)$ and dt_k equal to zero.

The simple expedient[4,5] of writing (1) as (2) and applying (5) has the same effect as (and is equivalent to) the expansion of variables to map all of the subintervals onto a single interval. The relative ease of this suggested alternative should be apparent.

Received April 2, 1965.

* Senior Analyst; also Consultant in Space Technology. Member AIAA.

† Numbers in parentheses refer to equations in the subject paper.

References

[1] Mason, J. D., Dickerson, W. B., and Smith, D. B., "A variational method for optimal staging," AIAA J. **3**, 2007–2012 (1965).

[2] Denbow, C. H., *A Generalized Form of the Problem of Bolza* (University of Chicago Press, Chicago, Ill., 1937).

[3] Hunt, R. W. and Andrus, J. F., "Optimization of trajectories having discontinuous state variables and intermediate boundary conditions," AIAA-IMS-SIAM-ONR Symposium on Control and System Optimization, Monterey, Calif. (January 1964).